Electric & Magnetic fields of a point charge in arbitrary motion

Using the Liénard-Wiechert potentials

$$\left.\begin{aligned} V(r,t) &= \frac{1}{4\pi\epsilon_0}\frac{qc}{(\imath c - \vec{\imath}\cdot\vec{v})} \\ A(r,t) &= \frac{v}{c^2}V(r,t) \end{aligned}\right\} \quad \text{--- (1)}$$

eq's for E and B

$$E = -\nabla V - \frac{\partial A}{\partial t}, \quad B = \nabla \times A$$

differentiation is tricky

$$\vec{\imath} = r - w(t_r) \quad \text{and} \quad v = \dot{w}(t_r) \quad \text{--- (2)}$$

are both evaluated at the retarded time, and t_r → defined implicitly by the eq^n

$$|r - w(t_r)| = c(t - t_r) \quad \text{--- (3)}$$

– is itself function of r and t.

Now

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{-1}{(\imath c - \vec{\imath}\cdot\vec{v})^2}\nabla(\imath c - \vec{\imath}\cdot\vec{v}) \quad \text{--- (4)}$$

Since $\imath = c(t - t_r)$

$$\nabla \imath = -c\nabla t_r \quad \text{--- (5)}$$

As for second term, product rule 4 gives

$$\nabla(\vec{\imath}\cdot\vec{v}) = (\vec{\imath}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{\imath} + \vec{\imath}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{\imath}) \quad \text{--- (6)}$$

Evaluating these terms one at a time.

$$(\vec{\imath}\cdot\nabla)v = \left(\imath_x\frac{\partial}{\partial x} + \imath_y\frac{\partial}{\partial y} + \imath_z\frac{\partial}{\partial z}\right)v(t_r)$$

$$= \imath_x\frac{dv}{dt_r}\frac{\partial t_r}{\partial x} + \imath_y\frac{dv}{dt_r}\frac{\partial t_r}{\partial y} + \imath_z\frac{dv}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= a(\vec{\imath}\cdot\nabla t_r) \quad -(7)$$

$a \equiv \dot{v}$ acceleration of the particle at retarded time.

Now $(v\cdot\nabla)\vec{\imath} = (v\cdot\nabla)r - (v\cdot\nabla)w \quad -(8)$

and $(v\cdot\nabla)r = \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z})$

$$= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} = v \quad -(9)$$

while $(v\cdot\nabla)w = v(v\cdot\nabla t_r)$ (same as eqn 7)

$$\nabla\times v = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

$$= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{z}$$

$$= -a\times\nabla t_r \quad -(10)$$

Finally $\nabla\times\vec{\imath} = \nabla\times r - \nabla\times w \quad -(11)$

$\underline{\nabla\times r = 0}$ while same as eqn (10)

$$\underline{\nabla\times w = -v\times\nabla t_r} \quad -(12)$$

Putting all this in eqn (6) and using the "BAC-CAB" rule to reduce the triple cross products

$$\nabla(\vec{r}\cdot\vec{v}) = a(\vec{r}\cdot\nabla t_r) + v - v(v\cdot\nabla t_r) - \vec{r}\times(a\times\nabla t_r) + v\times(v\times\nabla t_r)$$

$$= v + (\vec{r}\cdot a - v^2)\nabla t_r \quad \text{—(13)}$$

From eqⁿ (9) & (13)

$$\nabla V = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(rc-\vec{r}\cdot\vec{v})^2}\left[v + (c^2 - v^2 + \vec{r}\cdot a)\nabla t_r\right] \quad \text{—(14)}$$

We need to know ∇t_r

$$-c\nabla t_r = \nabla r = \nabla\sqrt{\vec{r}\cdot\vec{r}}$$

$$= \frac{1}{2\sqrt{\vec{r}\cdot\vec{r}}}\nabla(\vec{r}\cdot\vec{r})$$

$$= \frac{1}{r}\left[(\vec{r}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{r})\right] \quad \text{—(15)}$$

But $(\vec{r}\cdot\nabla)\vec{r} = \vec{r} - v(\vec{r}\cdot\nabla t_r)$ eqⁿ (11) & (12)

While $\nabla\times\vec{r} = (v\times\nabla t_r)$

Thus

$$-c\nabla t_r = \frac{1}{r}\left[\vec{r} - v(\vec{r}\cdot\nabla t_r) + \vec{r}\times(v\times\nabla t_r)\right]$$

$$= \frac{1}{r}\left[\vec{r} - (\vec{r}\cdot v)\nabla t_r\right]$$

hence $$\nabla t_r = \frac{-\vec{r}}{rc - \vec{r}\cdot\vec{v}} \quad \text{—(16)}$$

From eqⁿ (14) now

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(rc-\vec{r}\cdot\vec{v})^3}\left[(rc - \vec{r}\cdot\vec{v})v - (c^2 - v^2 + \vec{r}\cdot a)\vec{r}\right] \quad \text{—(17)}$$

Similar calculation yields

$$\frac{\partial A}{\partial t} = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(rc-\vec{r}\cdot\vec{v})^3}\left[(rc - \vec{r}\cdot\vec{v})(-v + ra/c) + \frac{r}{c}(c^2 - v^2 + \vec{r}\cdot\vec{a})v\right] \quad \text{—(18)}$$

Combining these results and introducing the vector

$$u \equiv c\hat{\mathscr{r}} - v \quad \text{---} (19)$$

$$E(r,t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot u)^3}\left[(c^2 - v^2)u + \vec{\mathscr{r}} \times (u \times a)\right] \quad \text{---} (20)$$

$$\nabla \times A = \frac{1}{c^2} \nabla \times (Vv)$$

$$= \frac{1}{c^2}\left[V(\nabla \times v) - v \times (\nabla V)\right]$$

Putting values of $\nabla \times v$ and ∇V from eqn (16) & (17)

$$\nabla \times A = -\frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{1}{(u\cdot\vec{\mathscr{r}})^3} \vec{\mathscr{r}} \times \left[(c^2 - v^2)v + (\vec{\mathscr{r}}\cdot\vec{a})v + (\vec{\mathscr{r}}\cdot u)\right].$$

$$\boxed{B(r,t) = \frac{1}{c}\hat{\mathscr{r}} \times E(r,t)} \quad \text{---} (21)$$

Magnetic field of a point charge is always perpendicular to the electric field and to the vector from the retarded point. The first term in E falls off as the inverse square of the distance from the particle. If the velocity & acceleration are both zero, this term only survives

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$

First term → generalized Coulomb field
↳ velocity field (does not depend on acceleration)

Second term → $\vec{\mathscr{r}} \times (u \times a)$ falls off as inverse first power of $\mathscr{r}$ and dominant at large distances → responsible for e.m. radiation → radiation field → acceleration field ($\propto a$)